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GLOBAL REGULARITY ON 3-DIMENSIONAL SOLVMANIFOLDS

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ABSTRACT. Let M be any 3-dimensional (nonabelian) compact solvmanifold. We apply the methods of representation theory to study the convergence of Fourier series of smooth global solutions to first order invariant partial differential equations $Df = g$ in $C^\infty(M)$. We show that smooth infinite-dimensional irreducible solutions, when they exist, satisfy estimates strong enough to guarantee uniform convergence of the irreducible (or primary) Fourier series to a smooth global solution.

1. INTRODUCTION

Let S be a solvable Lie group, and Γ a discrete subgroup of S with compact quotient $\Gamma\backslash S$. There is then a unique probability measure ν on $\Gamma\backslash S$ that is invariant under translation on the right by elements of S. The regular representation of S on $L^2(\Gamma\backslash S, \nu)$ decomposes into a direct sum of a countable number of irreducible unitary representations π of S, each of finite multiplicity m_π [G]. Let D be a *first order* differential operator with complex coefficients, left-invariant on S and viewed on $\Gamma\backslash S$. Let $(\Gamma\backslash S)\hat{}$ denote the dual object of $\Gamma\backslash S$. If $g \in C^\infty(\Gamma\backslash S)$ and if g_π is an orthogonal component of g corresponding to some irreducible unitary representation π, then $g_\pi \in C^\infty(\Gamma\backslash S)$ too [A-B]. Modulo unitary equivalence, we may think of g_π as being a C^∞-vector in any concrete realization, or model, of π. We are interested in algebraically well-defined conditions on D under which the global solvability of $Df = g$ in $C^\infty(\Gamma\backslash S)$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in the C^∞-vectors for each π in the spectrum of $\Gamma\backslash S$. In a sense, we are looking for algebraic conditions on D for the reduction of a global (geometrical) problem on $\Gamma\backslash S$ to a collection of purely group (representation) theoretic problems, none of which needs to be regarded as living on the manifold $\Gamma\backslash S$. Informally speaking, operators D admitting such a reduction are called *globally regular* (Definition (1.1)).

In order to describe the results, we will recall the classical situation on a torus T^2 of two dimensions (the situation being similar for T^n with $n > 2$). Let $D = \alpha\partial/\partial x + \beta\partial/\partial y$ and suppose for simplicity that α and β are real.

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Then D is globally regular if and only if β/α is not a (transcendental) Liouville number. The problem with Liouville numbers is that, in solving for the Fourier transform of the solution function, *small divisors* occur. Now, every solvmanifold $\Gamma\backslash S$ contains the structure of a torus $T = \Gamma[S, S]\backslash S$ of dimension ≥ 1, although this torus does not reflect any of the nonabelian structure of S. The only representations in $(\Gamma\backslash S)\hat{}$ which are not infinite dimensional are the one-dimensional characters of $\Gamma[S, S]\backslash S$. Since the presence of this torus is inescapable, we denote, for each $g \in C^\infty(\Gamma\backslash S)$, the sum of the one-dimensional components of g by g_0. Then global regularity is defined as follows. Let $L^2(\Gamma\backslash S) = \bigoplus_{\pi\in(\Gamma\backslash S)\hat{}, j=1,\dots,m_\pi} H_{\pi,j}$ be any (noncanonical) irreducible decomposition of $L^2(\Gamma\backslash S)$.

(1.1) **Definition.** *A left-invariant differential operator D on $\Gamma\backslash S$ is called* globally regular *if the three conditions*

(1) $g \in C^\infty(\Gamma\backslash S)$,
(2) *For each $\pi \in (\Gamma\backslash S)\hat{}$ and $\forall j = 1, \dots, m_\pi \exists$ a solution in $C^\infty(\Gamma\backslash S)$ to $Df_{\pi,j} = g_{\pi,j}$ (where $g_{\pi,j} = (\pi, j)$-component of g), and*
(3) $\exists f_0 \in C^\infty(T)$ *such that* $Df_0 = g_0$,

imply that there exists a solution in $C^\infty(\Gamma\backslash S)$ to $Df = g$.

Note that the solutions in (2) could be found in any convenient realization of π.

In previous papers we have dealt with *nilpotent* S. On the simplest nilmanifolds, the 3-dimensional Heisenberg manifolds, every first order differential operator D in the complexified Lie algebra is globally regular [R2]. On more complicated nilmanifolds the problem of small divisors arises in the representation spaces of the group as well as on the associated torus. Moreover, if $D = X + iY$ is regular, both ad_X and ad_Y must map each step of the lower central series of the Lie algebra of S (nilpotent) onto a sufficiently large subset of the next step. The details are explained in [C-R1, p. 349]. The purpose of this paper is to investigate the global regularity of first order differential operators on 3-dimensional compact *solvmanifolds.* We show that, as in the case of the simplest nilmanifolds, every first order differential operator on a 3-dimensional compact solvmanifold is globally regular.

2. 3-DIMENSIONAL SOLVMANIFOLDS

All 3-dimensional compact solvmanifolds can be described (up to homeomorphism) as quotients of two groups S_h and S_r by their various cocompact discrete subgroups. The groups S_h and S_r can both be described as $R^2 \rtimes R^1$, where $(x, t)(x', t') = (x + A^t x', t + t')$. Here A^t is a 1-parameter subgroup of $SL(2, R)$ through a matrix $A \in SL(2, Z)$. S_h arises when the eigenvalues of A are $\lambda > 1$ and λ^{-1}, so that the orbits of R^1 in R^2 are *hyperbolic.* S_r arises when A^t is a compact group of *rotations* of R^2. Let $N := R^2 \times \{0\}$, the (abelian) nilradical of S. The cocompact discrete subgroups Γ are described in [A-G-H], based upon the facts (due to Mostow) that $\Gamma \cap N$ is a discrete lattice in R^2, and that the image of Γ under the natural projection $S \to S/N$ is a discrete lattice in R^1. We remark that $\Gamma\backslash S_h$ is determined up to homeomorphism by the eigenvalue $\lambda > 1$ of A, and λ must be such that $\lambda + \lambda^{-1} \in Z$. For this reason, we denote the 'hyperbolic' manifolds $\Gamma_\lambda\backslash S_h$. Note however that

S_h is independent of the value of $\lambda > 1$. We have good use for the following lemma ((3.4) in [C-R2]).

(2.1) **Lemma.** *If $S_h = R^2 \rtimes R^1$ with the diagonalized matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and if Γ_λ is a cocompact discrete subgroup of S_h, then $\Gamma_\lambda \cap N$ is an abelian lattice $\mathscr{L}$ of points (α, β) having the property that the product $\alpha\beta$ is bounded away from zero, except of course at the identity.*

We need also the following

(2.2) **Corollary.** *In the setting of the lemma above, the dual lattice $\mathscr{L}^* = \{\chi_{(a,b)} : \mathscr{L} \to 1\}$ is also a lattice of points (a, b) such that the product ab is bounded away from 0 (except of course for $(a, b) = (0, 0)$).*

This corollary will give useful information about $(\Gamma\backslash S)^\wedge_\infty$, the infinite dimensional representations in the spectrum of $\Gamma\backslash S$, in the hyperbolic case. For *all* 3-dimensional compact solvmanifolds, $(\Gamma\backslash S)^\wedge_\infty$ is constructed as follows. Let $\chi_{(\alpha,\beta)} \in \hat{N}$, where $\chi_{(\alpha,\beta)} : \Gamma \cap N \to 1$. Now let M be the extension of N by the stabilizer of $\chi_{(\alpha,\beta)}$ in S, and extend χ to M, so $\pi_{\alpha\beta} := \mathrm{Ind}_M^S(\chi_{(\alpha,\beta)}) \in \hat{S}$. If $S = S_h$ then $M = N$ and $M\backslash S \cong R$, whereas if $S = S_r$ then $M \cong N \times Z$ and $M\backslash S \cong R/Z = \textit{the circle group}$. If $H_{(\chi,M)}$ is the standard Mackey induced representation space, then $H_{(\chi,M)} = \{ f : S \to C | f(ms) = \chi_{(\alpha,\beta)}(m)f(s), |f| \in L^2(M\backslash S) \}$. Define $L : H_{(\chi,M)} \to L^2(\Gamma\backslash S)$ by $(Lf)(\Gamma s) = \sum_{\Gamma\cap M\backslash\Gamma} f(\gamma s)$. Then L is a right S-invariant injection. If $\mathrm{Int}(\chi, M) = \{t \in R | \chi^{\exp tT} : \Gamma \cap M \to 1\}$ where $\chi^a(b) = \chi(a^{-1}ba)$ then the multiplicity of $\pi_{\alpha\beta}$ in $L^2(\Gamma\backslash S)$ equals the number of distinct Γ-orbits in $\mathrm{Int}(\chi, M)$.

In the case of $S = S_h$, it is easiest to describe $\Gamma_\lambda\backslash S_h$ if we take $\Gamma_\lambda \cap N = Z^2$ and $A \in SL(2, Z)$. However, the model for $\pi_{\alpha\beta}$ is simplest if A is diagonalized ($A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ as in Lemma (2.1)) and then $\Gamma \cap N$ is more difficult to describe. Nevertheless, the Corollary (2.2) above shows that in this model $\{\pi_{\alpha\beta} \in (\Gamma_\lambda\backslash S_h)^\wedge_\infty\}$ has $\alpha\beta$ *bounded away* from zero. We remark upon the fact that $\alpha\beta \neq 0$ prevents $\pi_{\alpha\beta}$ from being a representation of the (less well-behaved) '$ax + b$' group which is a quotient of S_h.

We need to say a few words about the rotational three dimensional solvmanifolds $\Gamma\backslash S_r$ as well. Unlike the hyperbolic case, there are only finitely many $\Gamma\backslash S_r$ up to homeomorphism. We take $A \in SL(2, Z)$, but this time with *no* eigenvalues > 1. Now A turns out to be similar to a *rotation* by $2\pi/p$, where $p = 1, 2, 3, 4$, or 6 (see [A-G-H]). Since S_r is independent of p (up to group isomorphism), we denote the distinct rotational three dimensional solvmanifolds by $\Gamma_p\backslash S_r$, $p = 1$, 2, 3, 4, or 6.

Our main result is

(2.3) **Theorem.** *Let $\Gamma\backslash S$ be any nonabelian 3-dimensional compact solvmanifold. If $D \in \mathscr{S}^C$, the complexified Lie algebra of S, then D is globally regular on $\Gamma\backslash S$.*

Since this result has been proved in an earlier paper when S is nilpotent, we concern ourselves here only with the manifolds $\Gamma_\lambda\backslash S_h$ and $\Gamma_p\backslash S_r$. Note also that the associated torus T is 1-dimensional, so that small divisors *cannot* occur on T. Thus condition (3) of Definition (1.1) will be satisfied automatically whenever (2) holds.

We will divide our proof of the theorem into two sections, one dealing with the rotational group S_r, the other with the hyperbolic one, S_h.

3. Proof of the Theorem—case of $\Gamma_p \backslash S_r$

Let $\mathscr{S}_r$ be the Lie algebra of S_r with a linear basis T, X, Y and the commutation relations $[T, X] = -\frac{2\pi}{p} Y$ and $[T, Y] = \frac{2\pi}{p} X$. Let $\pi_{\alpha\beta}$ be a generic infinite-dimensional representation in $(\Gamma_p \backslash S_r)\hat{}$, acting in $L^2(T)$, where $T = M \backslash S_r$, a 1-dimensional torus as described in §2. For $f \in L^2(T)$ the action of $\pi_{\alpha\beta}$ on f is given by

$$\pi_{\alpha\beta}(x, y; t) f(\tau) = \exp(2\pi i \langle (\alpha, \beta)\sigma(\tfrac{2\pi}{p}\tau), (x, y)\rangle) f(\tau + t),$$

where $\sigma(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$. For the basis vector fields X, Y, T this amounts to

$$\begin{aligned} d\pi_{\alpha\beta}(X) &= 2\pi i \left(\alpha \cos \frac{2\pi}{p} t - \beta \sin \frac{2\pi}{p} t \right), \\ d\pi_{\alpha\beta}(Y) &= 2\pi i \left(\alpha \sin \frac{2\pi}{p} t + \beta \cos \frac{2\pi}{p} t \right), \\ d\pi_{\alpha\beta}(T) &= \frac{d}{dt}. \end{aligned} \tag{3.1}$$

Since the constant p plays a negligible role in the proof (even though it classifies the rotational 3-dimensional solvmanifolds), we will set $p = 1$ in what follows.

We will break the proof into two cases of $D \in \mathscr{S}_r^C$.

Case 1. $D = X + \gamma Y$, $\gamma \in C$. This essentially covers all $D \in \mathscr{N}^C$ (up to a constant factor) since $D = Y$ and $D = X$ behave alike and the case of $D = 0$ is trivial.

Case 2. $D = T + i(aX + bY)$, $a, b \in R$. This covers all $D \in \mathscr{S}_r^C \sim \mathscr{N}^C$ (up to an isomorphism) because the real part of D can be absorbed into T.

Proof of Case 1. Write $D = X + (a + ib)Y$, $a, b \in R$. Then, in view of (3.1), the operator $d\pi_{\alpha\beta}(D)$ is a multiplication by the function

$$\begin{aligned} D_{\alpha\beta}(t) :&= 2\pi i[(\alpha + a\beta + ib\beta)\cos 2\pi t + (a\alpha - \beta + ib\alpha)\sin 2\pi t] \\ &= 2\pi i(w \cos 2\pi t + z \sin 2\pi t). \end{aligned} \tag{3.2}$$

By hypothesis, the equation $d\pi_{\alpha\beta}(D) f_{\alpha\beta} = g_{\alpha\beta}$ has the solution

$$f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/D_{\alpha\beta}(t). \tag{3.3}$$

To prove the theorem we need to show that $\sum_{(\alpha, \beta)} L f_{\alpha\beta} \in C^\infty(\Gamma \backslash S)$. Here (α, β) varies over a cross section of Γ-orbits, so that each infinite-dimensional primary summand will be spanned. By the Auslander-Brezin version of the Sobolev inequality it suffices to show $\sum_{(\alpha, \beta)} \|U L f_{\alpha\beta}\|^2 < \infty$ for all $U \in \mathscr{U}(\mathscr{S}_r)$. Since L is an S-invariant isometry from $H_{\alpha\beta}$ into $L^2(\Gamma \backslash S)$, this is the same as to show $\sum_{(\alpha, \beta)} \|U f_{\alpha\beta}\|_2^2 < \infty$. We begin by estimating the sum

$\sum \|f_{\alpha\beta}\|_2^2$. The problem, of course, is that $D_{\alpha\beta}(t)$ in (3.3) can have zeros, and that even when it has no zeros, we must know how close $|D_{\alpha\beta}(t)|$ can come to 0 as $(\alpha, \beta) \to \infty$. We can write

$$D_{\alpha\beta}(t) = \pi i((w - iz)e^{2\pi it} + (w + iz)e^{-2\pi it}) = \pi i(Ae^{2\pi it} + Be^{-2\pi it}),$$

with A and B defined by the last equation. The minimum of $|D_{\alpha\beta}(t)|$ occurs where A and B are rotated to opposite directions, and then

$$\begin{aligned} \operatorname{Min} |D_{\alpha\beta}(t)| &= \pm\pi(|A| - |B|) \\ &= \pi(\alpha^2 + \beta^2)^{1/2}((a^2 + (1+b)^2)^{1/2} - (a^2 + (1-b)^2)^{1/2}) \\ &= (\alpha^2 + \beta^2)^{1/2} \cdot K \geq \sqrt{2}|\alpha\beta|^{1/2} \cdot K. \end{aligned} \tag{3.4}$$

If $b \neq 0$, the constant K is $\neq 0$. Since $|\alpha\beta| \gg 0$, $|f_{\alpha\beta}(t)| \leq C|g_{\alpha\beta}(t)|$ with the constant C independent of (α, β). Consequently, $\sum_{(\alpha,\beta)} \|f_{\alpha\beta}\|^2 \leq C\sum_{(\alpha,\beta)} \|g_{\alpha\beta}\|^2 < \infty$ for $g \in C^\infty(\Gamma\backslash S)$.

If $b = 0$, $D_{\alpha\beta}(t)$ does have one or more zeros and

$$D_{\alpha\beta}(t) = 2\pi i((\alpha + a\beta)\cos 2\pi t + (a\alpha - \beta)\sin 2\pi t).$$

If $D_{\alpha\beta}(t_0) = 0$, then $g_{\alpha\beta}(t_0) = 0$ too, since $f_{\alpha\beta}$ given by (3.3) is in $C^\infty(T)$. The idea is to control $f_{\alpha\beta}$ inside specified intervals around each of the t_0's by the mean value theorem, and to use the monotonicity of $D_{\alpha\beta}$ on large exterior intervals to control $|f_{\alpha\beta}|$ by keeping $|D_{\alpha\beta}|$ big. We have

$$D'_{\alpha\beta}(t_0) = -2\pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2},$$

and by the mean value theorem

$$|D'_{\alpha\beta}(t) - D'_{\alpha\beta}(t_0)| \leq |t - t_0| 4\pi^2(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}.$$

Consequently, for $|t - t_0| < 1/4\pi$, $|D'_{\alpha\beta}(t)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}$ and, since $D_{\alpha\beta}(t_0) = 0$,

$$|D_{\alpha\beta}(t)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}|t - t_0|$$

again by the mean value theorem. For $g_{\alpha\beta}(t)$ we have the estimate

$$|g_{\alpha\beta}(t)| = |g_{\alpha\beta}(t) - g_{\alpha\beta}(t_0)| \leq \|Tg_{\alpha\beta}\|_\infty |t - t_0|, \tag{3.5}$$

$\|\cdot\|_\infty$ denoting the sup norm on the torus $M\backslash S$. So

$$|f_{\alpha\beta}(t)| \leq \|Tg_{\alpha\beta}\|_\infty \tfrac{1}{\pi}(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2}, \quad \text{for } |t - t_0| < 1/4\pi. \tag{3.6}$$

On the intervals complementary to $|t - t_0| < 1/4\pi$,

$$|D_{\alpha\beta}(t)| \geq |D_{\alpha\beta}(t_0 \pm 1/4\pi)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}/4\pi,$$

hence

$$|f_{\alpha\beta}(t)| \leq 4\|g_{\alpha\beta}\|_\infty(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2} \quad \text{for } |t - t_0| \geq 1/4\pi \tag{3.7}$$

for each of the two values t_0 of t where $D_{\alpha\beta}$ vanishes. By (3.6) and (3.7), for all t on the torus $M\backslash S$ we have the following estimate:

$$|f_{\alpha\beta}(t)| \leq (\|Tg_{\alpha\beta}\|_\infty/\pi + 4\|g_{\alpha\beta}\|_\infty)(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2}. \tag{3.8}$$

By Sobolev's inequality we may replace the sup norms in (3.8) by L^2-norms of $g_{\alpha\beta}$, $Tg_{\alpha\beta}$, and $T^2g_{\alpha\beta}$. The sum $\sum_{(\alpha,\beta)} \|f_{\alpha\beta}\|_2^2$ is finite because each sum $\sum_{(\alpha,\beta)} \|T^k g_{\alpha\beta}\|_2^2$ for $k = 0, 1, 2$ is finite and $|\alpha\beta|$ is bounded away from zero.

Next, we must show $\sum_{(\alpha,\beta)} \|U f_{\alpha\beta}\|_2^2 < \infty$ for every fixed $U \in \mathscr{U}(\mathscr{S}_r)$. Since $[D, \mathscr{N}] = 0$, this is true for all $U \in \mathscr{U}(\mathscr{N})$. It remains to show $\sum_{(\alpha,\beta)} \|T^k f_{\alpha\beta}\|_2^2 < \infty$ for $k = 1, 2, \ldots$, because every $U \in \mathscr{U}(\mathscr{S}_r)$ can be written as a linear combination of monomials $T^k V$ with V in $\mathscr{U}(\mathscr{N})$.

(3.9) **Proposition.** *For $D = X+\gamma Y$, $\gamma \in C$ and $k = 1, 2, 3, \ldots$ the $(k+1)$-fold bracket product $[D[D\cdots[D, T^k]\cdots]]$ is* 0.

Proof of Proposition. By Leibnitz's rule for the derivation $[D, \cdot]$ of the algebra $\mathscr{U}(\mathscr{S}_r)$ we have $[D, T^k] = \sum_{j=1}^k T\cdots T[D, T]T\cdots T$, with $[D, T]$ at jth place. Since D commutes with $\mathscr{N}$ and $[D, T] = -X+\gamma Y \in \mathscr{N}$, the derivation repeated $k+1$ times is zero.

(3.10) **Proposition.** *For $f_{\alpha\beta}$ as in* (3.3) *we have*

$$T^k f_{\alpha\beta} = h_k / D_{\alpha\beta}^{k+1} \tag{3.11}$$

with

$$h_k = [D[D\cdots[D, T^k]\cdots]]g_{\alpha\beta} + D[D\cdots[D, T^k]\cdots]g_{\alpha\beta} + \cdots + D^{k-1}[D, T^k] + D^k T^k g_{\alpha\beta}.$$

The first bracket involves k D's with the number of D's inside the brackets decreasing by one in each successive summand.

Proof of Proposition. In view of Proposition (3.9) this is formula (1.8) on p. 353 of [C-R1].

The estimates on $T^k f_{\alpha\beta}$ given by (3.11) can now be done in a manner similar to that already presented. If $D_{\alpha\beta} \neq 0$, we use the inequality (3.4) raised to the power $k+1$ to estimate the denominator $|D_{\alpha\beta}|^{k+1}$. If $D_{\alpha\beta}(t_0) = 0$, the numerator h_k in (3.11) must have a $(k+1)$th order zero at t_0 since $T^k f_{\alpha\beta}$ is C^∞. Instead of the estimate (3.5) we use

$$|h_k(t)| \leq \|T^{k+1} h_k\|_\infty |t - t_0|^{k+1}/(k+1)!$$

which follows from Taylor's formula. In the denominator of (3.6) and (3.7) we use the $(k+1)$th power of the previous estimate for $D_{\alpha\beta}$.

Proof of Case 2. Here $D = T + i(aX + bY)$, $a, b \in R$, and

$$\begin{aligned} d\pi_{\alpha\beta}(D) &= \frac{d}{dt} - 2\pi((a\alpha + b\beta)\cos 2\pi t + (b\alpha - a\beta)\sin 2\pi t) \\ &= \frac{d}{dt} - 2\pi(A\cos 2\pi t + B\sin 2\pi t). \end{aligned} \tag{3.12}$$

Write $A\cos 2\pi t + B\sin 2\pi t = (A^2 + B^2)^{1/2}\sin 2\pi(t + \phi)$, with the constant ϕ depending upon A and B. If $d\pi_{\alpha\beta}(D)f_{\alpha\beta} = g_{\alpha\beta}$, we have

$$f_{\alpha\beta}(t) = \exp(-2\pi(A^2 + B^2)^{1/2}\cos 2\pi(t + \phi)) \times \left(\int_{-\phi-1/2}^{t} g_{\alpha\beta}(\tau)\exp(2\pi(A^2 + B^2)^{1/2}\cos 2\pi(\tau + \phi))\,d\tau + C\right).$$

Here we identify $M\backslash S = R/Z$ with the interval $[-\phi - \frac{1}{2}, \frac{1}{2} - \phi]$, and we use the fact that $f_{\alpha\beta}$, $g_{\alpha\beta}$, and the exponentials all have period 1. Since C is arbitrary, choosing $C = 0$ and changing the variables $\tau' = \tau + \phi$, $t' = t + \phi$ we have

$$f_{\alpha\beta}(t-\phi) = \int_{-1/2}^{t} g(\tau-\phi)\exp(2\pi(A^2+B^2)^{1/2}(\cos 2\pi\tau - \cos 2\pi t))\,d\tau$$
$$= -\int_{t}^{1/2} \quad \text{(same integrand as above).}$$

The last equality follows from the periodicity of $f_{\alpha\beta}$, because $0 = f_{\alpha\beta}(-1/2-\phi) = f_{\alpha\beta}(1/2-\phi)$. For the estimates on $f_{\alpha\beta}(t)$ or on $f_{\alpha\beta}(t-\phi)$ we use the integral over $-1/2 \le \tau \le t$ if $-1/2 \le t \le 0$ and over $t \le \tau \le 1/2$ for $0 < t < 1/2$. That way we always have $\cos 2\pi\tau < \cos 2\pi t$, so that the exponent in the integral defining $f_{\alpha\beta}$ is negative. Then

$$\|f_{\alpha\beta}\|_\infty \le C\|g_{\alpha\beta}\|_\infty \le C_1(\|g_{\alpha\beta}\|_2 + \|Tg_{\alpha\beta}\|_2)$$

Since the $f_{\alpha\beta}$'s all live on an interval of length 1, the same inequality holds for $\|f_{\alpha\beta}\|_2$ with a constant, say C_1', independent of (α, β). Thus $\sum_{(\alpha,\beta)}\|f_{\alpha\beta}\|_2^2 < \infty$ and $f = \sum f_{\alpha\beta} \in L^2(\Gamma_p\backslash S_r)$. Next, to estimate $\sum_{(\alpha,\beta)}\|Uf_{\alpha\beta}\|_2^2$ for every $U \in \mathscr{U}(\mathscr{S}_r)$ it suffices to consider $U = X^kY^jT^l$. By (3.12) T just differentiates $f_{\alpha\beta}$ yielding $2\pi(A^2+B^2)^{1/2}\sin 2\pi(t+\phi)f_{\alpha\beta}(t) + g_{\alpha\beta}(t)$ with A, B depending linearly on α, β. Successive powers of T differentiate $g_{\alpha\beta}$ and $\sin 2\pi(t+\phi)$, or $f_{\alpha\beta}(t)$. Operating by X's or Y's just multiplies by a polynomial in (α, β) times a (bounded) combination of sines and cosines. However, X^2+Y^2 acts on $L^2(M\backslash S)$ by multiplying by $-4\pi^2(\alpha^2+\beta^2)$, so that $\sum_{(\alpha,\beta)}(\alpha^2+\beta^2)^q\|g_{\alpha\beta}\|_2^2 < \infty$. Thus $\sum_{(\alpha,\beta)}\|X^kY^jT^lf_{\alpha\beta}\|_2^2$ can be estimated by $C\sum\|(X^2+Y^2)^q g_{\alpha\beta}\|_2^2$ for some q.

Remark. Although we have pretended $p = 1$ in the arguments above, this was just a notational convenience, and the same arguments work just as well for general $\Gamma\backslash S_r$. Actually, the use of $\Gamma_1\backslash S_r$ has some independent interest however. Here, although S_r is *not* nilpotent, $\Gamma \cong Z^3$. Thus, by the general theory of Mostow [M], $\Gamma_1\backslash S_r$ is homeomorphic to a torus T^3. Here we get a proof of global regularity for elements of $\mathscr{S}_r^C$ acting on T^3, keeping in mind that we mean global regularity with respect to the representation theory of S_r. Thus we have a natural class of operators on T^3 which are best analyzed with respect to S_r.

4. Proof of the theorem—case of $\Gamma_\lambda\backslash S_h$

Let $\mathscr{S}_h$ be the Lie algebra of S_h with a linear basis T, X, Y and the commutation relations $[T, X] = X\ln\lambda$ and $[T, Y] = -Y\ln\lambda$. A generic infinite dimensional representation $\pi_{\alpha\beta}$ in $(\Gamma_\lambda\backslash S_h)\hat{}$ acts on $L^2(R)$ by

$$\pi_{\alpha\beta}(x, y, t)f(\tau) = \exp 2\pi i(\alpha\lambda^\tau x + \beta\lambda^{-\tau}y)f(\tau+t).$$

For the basis vector fields X, Y, T this amounts to

$$d\pi_{\alpha\beta}(X) = 2\pi i\alpha\lambda^t, \quad d\pi_{\alpha\beta}(Y) = 2\pi i\beta\lambda^{-t}, \quad d\pi_{\alpha\beta}(T) = \frac{d}{dt}. \tag{4.1}$$

The space $H_{\alpha\beta}^\infty$ of C^∞-vectors for $\pi_{\alpha\beta}$ consists of $C^\infty(R)$ functions ϕ with

$$\lim_{|t|\to\infty} \lambda^{m|t|}\phi^{(n)}(t) = 0$$

for every $m, n \in N$. These ϕ are called "super-Schwartz". We will denote this space by $\mathscr{S}\mathscr{S}(R)$.

We will break the proof into four cases of $D \in \mathscr{S}_h^C$.

Case 1. $D = T$. Any nonzero complex multiple of $D = T + aX + bY \in \mathscr{S}_h$ can be reduced up to isomorphism to $D = T$, if $a, b \in R$.

Case 2. $D = aX + bY \in \mathscr{N} \subset \mathscr{S}_h$, $a, b \in R$.

Case 3. $D = aX + bY \in \mathscr{N}^C \subset \mathscr{S}_h^C$, $a, b \in C$.

Case 4. $D = T + i(aX + bY)$, $a, b \in R$.

Proof of Case 1. $D = T$. The equation $d\pi_{\alpha\beta}(T)f_{\alpha\beta} = g_{\alpha\beta}$ becomes $(d/dt)f_{\alpha\beta} = g_{\alpha\beta}$. Since both $f_{\alpha\beta}$ and $g_{\alpha\beta}$ are in $H_{\alpha\beta}^\infty$, we have

$$f_{\alpha\beta}(t) = \int_{-\infty}^{t} g_{\alpha\beta}(\tau)\,d\tau = -\int_{t}^{\infty} g_{\alpha\beta}(\tau)\,d\tau. \tag{4.2}$$

Subcase 1a. If $t \geq 0$, we use the second integral for the estimates on $f_{\alpha\beta}$ and the fact that $g_{\alpha\beta}(\tau) = Xg_{\alpha\beta}(\tau)/2\pi i\alpha\lambda^\tau$, so that

$$|f_{\alpha\beta}(t)| \leq \|Xg_{\alpha\beta}\|_\infty (2\pi\alpha \ln\lambda)^{-1}\lambda^{-t} \in L^2([0, \infty), dt). \tag{4.3}$$

Squaring, integrating $\int_0^\infty \cdots dt$, and applying Sobolev's inequality to estimate the norm $\|Xg_{\alpha\beta}\|_\infty$ by a combination of L^2-norms of derivatives of $Xg_{\alpha\beta}$ we obtain

$$\|f_{\alpha\beta}\|^2_{L^2[0,\infty)} \leq \frac{c}{\alpha}(\|Xg_{\alpha\beta}\|_2^2 + \|TXg_{\alpha\beta}\|_2^2), \tag{4.4}$$

for some constant c. As in the proof of Lemma (2.1) (see [C-R2]), we pick the representative (α, β) from the orbit corresponding to π so that $\lambda^{-2} \leq |\beta/\alpha| \leq 1$. Since $|\alpha\beta| \gg 0$, we must have $|\alpha| \gg 0$ too. Thus $\sum_{(\alpha,\beta)} \|f_{\alpha\beta}\|^2_{L^2[0,\infty)} < \infty$.

Subcase 1b. If $t < 0$, we use the first integral formula for $f_{\alpha\beta}$ in (4.2), together with the fact that $g_{\alpha\beta}(\tau) = Yg_{\alpha\beta}(\tau)/2\pi i\beta\lambda^{-\tau}$. Picking (α, β) as in Subcase 1a, we insure that $|\beta| \gg 0$. Now an argument almost identical to that in Subcase 1a shows that $f = \sum f_{\alpha\beta} \in L^2(R)$.

To complete Case 1, we need to prove that $\sum \|Uf_{\alpha\beta}\|_2^2 < \infty$, for each fixed $U \in \mathscr{U}(\mathscr{S}_h)$. If $U = T^p$ there is no problem, since $[D, U] = 0$. If $U = X^mY^nT^p$, $Uf_{\alpha\beta}(t) = (2\pi i)^{m+n}\alpha^m\beta^n\lambda^{(m-n)t}T^{p-1}g_{\alpha\beta}(t)$. If $p \geq 1$, there is no problem. But if $p = 0$, we estimate $|g_{\alpha\beta}(\tau)|$ by a $\|Vg_{\alpha\beta}\|\lambda^{kt}$, where $V \in \mathscr{U}(\mathscr{S}_h)$ and $k \in Z$ are chosen one way for $t < 0$ and another way for $t \geq 0$, so as to assure that $Uf_{\alpha\beta} \in L^2(-\infty, 0)$ and $Uf_{\alpha\beta} \in L^2(0, \infty)$ separately. Hence $\sum Uf_{\alpha\beta} \in L^2(R)$.

Case 2. Suppose $D = aX + bY$, with a and b real. Here,

$$f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/2\pi i(a\alpha\lambda^t + b\beta\lambda^{-t}).$$

We will need to consider the occurrence of zeros in the denominator. Observe that the denominator vanishes if and only if $\lambda^{2t} = -b\beta/a\alpha$, which means $t = t_0 = \frac{1}{2}\log_\lambda(-b\beta/a\alpha)$.

Subcase 2a. Suppose $b\beta/a\alpha > 0$, so that $h(t) = (a\alpha\lambda^t + b\beta\lambda^{-t})^{-1}$ has a maximum (absolute) value on R. At the maximum point $h'(t) = 0$, so that

$\lambda^{2t} = b\beta/a\alpha$. Thus $h(t_{\max}) = \lambda^{t_{\max}}/2b\beta = \pm 1/2|ab\alpha\beta|^{1/2}$. Thus $|f_{\alpha\beta}(t)| \leq |g_{\alpha\beta}(t)|/2|ab\alpha\beta|^{1/2}$. Since $|\alpha\beta| \gg 0$, we have $\sum f_{\alpha\beta} \in L^2$, where the sum is over those (α, β) such that $b\beta/a\alpha > 0$. The same is true for $\sum U f_{\alpha\beta}$, for each fixed $U \in \mathscr{U}(\mathscr{S}_h)$ since

(1) if $U \in \mathscr{U}(\mathscr{N})$, $[U, D] = 0$, and
(2) if $U = X^m Y^n T^k$, then

$$\frac{d}{dt}(g_{\alpha\beta}(t)h(t)) = g'_{\alpha\beta}h + g_{\alpha\beta}h' = (Tg_{\alpha\beta})h + g_{\alpha\beta}h',$$

and $g_{\alpha\beta}h' = (X - Y)g_{\alpha\beta} \cdot h^2_{\alpha\beta}$, which is still L^2-summable.

Subcase 2b. Either $a\alpha = 0$ or $b\beta = 0$. Since $|\alpha\beta| \gg 0$ (except at $(0, 0)$) this means $a = 0$ or $b = 0$ (but *not* both). If $b = 0$, then $f_{\alpha\beta}(t) = -Yg_{\alpha\beta}(t)/4\pi^2 a\alpha\beta$. Then $\sum f_{\alpha\beta} \in L^2$, since $|\alpha\beta| \gg 0$. A similar argument applies if $a = 0$.

Subcase 2c. Here, we suppose there exists a zero, $t_0 = \frac{1}{2}\log_\lambda(-b\beta/a\alpha)$. Then $g_{\alpha\beta}(t_0) = 0$ too, and $f_{\alpha\beta}(t) = (g_{\alpha\beta}(t) - g_{\alpha\beta}(t_0))/2\pi i(a\alpha\lambda^t + b\beta\lambda^{-t})$. Since it would suffice to prove separately the summability of the real and imaginary parts of $f_{\alpha\beta}$, we can assume wlog that $g_{\alpha\beta}$ is *real*-valued. For some τ between 0 and t, we have $f_{\alpha\beta}(t) = g'_{\alpha\beta}(\tau)/2\pi i(a\alpha\lambda^\tau - b\beta\lambda^{-\tau})\ln\lambda$, by the general mean value theorem. Since $b\beta/a\alpha < 0$ in this case, the *new* denominator *cannot* vanish. Hence, recalling that τ depends on t,

$$\begin{aligned} 4\pi^2 &\int_R |f_{\alpha\beta}(t)|^2\,dt \\ &= \int_{|t-t_0|\leq 1} |g'_{\alpha\beta}(\tau)|^2 |a\alpha\lambda^\tau - b\beta\lambda^{-\tau}|^{-2} |\ln\lambda|^{-2}\,dt \\ &\quad + \int_{|t-t_0|\geq 1} |g_{\alpha\beta}(t)|^2 |a\alpha\lambda^t + b\beta\lambda^{-t}|^{-2}\,dt \\ &\leq \frac{\|g'_{\alpha\beta}\|^2_\infty}{-4a\alpha b\beta\ln^2\lambda} + \int_{|t-t_0|\geq 1} |g_{\alpha\beta}(t)|^2 \cdot \operatorname{Max}|a\alpha\lambda^t + b\beta\lambda^{-t}|^{-2}\,dt. \end{aligned}$$

The first summand is summable over (α, β), by Sobolev's inequality. However, by the mean value theorem, $|a\alpha\lambda^t + b\beta\lambda^{-t}| = |(a\alpha\lambda^\tau - b\beta\lambda^{-\tau})(\ln\lambda)(t - t_0)| \geq 2(-a\alpha b\beta)^{1/2}|\ln\lambda|$, if $|t - t_0| \geq 1$. Thus

$$\pi^2 \int_R |f_{\alpha\beta}(t)|^2\,dt \leq \frac{\|g'_{\alpha\beta}\|^2_\infty}{|a\alpha b\beta|\ln^2\lambda} + \int_R \frac{|g_{\alpha\beta}(t)|^2}{|a\alpha b\beta|\ln^2\lambda}\,dt,$$

and so $\sum f_{\alpha\beta} \in L^2$.

If $U \in \mathscr{U}(\mathscr{N})$, then $\sum U f_{\alpha\beta} \in L^2$ too, since $[D, U] = 0$. So it suffices to consider $\sum T^k f_{\alpha\beta}$, for each fixed $k \in Z^+$. First, consider $k = 1$. $2\pi i T f_{\alpha\beta}(t) = h(t)(a\alpha\lambda^t + b\beta\lambda^{-t})^{-2}$, where $h(t) = (a\alpha\lambda^t + b\beta\lambda^{-t})g'_{\alpha\beta}(t) - g_{\alpha\beta}(t)(a\alpha\lambda^t - b\beta\lambda^{-t})\ln\lambda$. Here $h(t)$ must have a zero of order at least two at $t = t_0$. Therefore, using a Taylor remainder of degree two in $(t - t_0)$, there exist τ and τ' such that $Tf_{\alpha\beta}(t) = h''(\tau')/D''(\tau)2\pi i$ where $D(t) = (a\alpha\lambda^t + b\beta\lambda^{-t})\ln\lambda$. But

$$|D''(t)| = 2([(a\alpha\lambda^\tau - b\beta\lambda^{-\tau})\ln\lambda]^2 + [(a\alpha\lambda^\tau + b\beta\lambda^{-\tau})\ln\lambda]^2) \geq 8|a\alpha b\beta|\ln^2\lambda,$$

since the second square is positive. Thus $|Tf_{\alpha\beta}(t)| \leq |h''(\tau')|/16\pi|a\alpha b\beta|$. Hence

$$4\pi^2 \int_R |Tf_{\alpha\beta}|^2 \leq \int_{|t-t_0|\leq 1} \|h''\|_\infty/64(a\alpha b\beta)^2\, dt + \int_R |h(t)|^2/\mathrm{Min}\{\, |D(t)|^2 \mid |t-t_0| \geq 1\,\}\, dt.$$

But $\|h''\|_\infty$ is bounded by norms of derivatives of $g_{\alpha\beta}$, providing good summability over (α, β) of the first term. Also, $|D(t)|^2 = |D''(\tau)(t-t_0)^2/2!|^2 \geq 16(a\alpha b\beta)^2 \ln^4 \lambda \gg 0$. Thus $\sum Tf_{\alpha\beta} \in L^2$.

Next, we consider $k > 1$. We make the following observations.

$$[D, T^k] = \sum_{j=1}^{k} T \cdots T[D, T]T \cdots T, \tag{4.5}$$

where $[D, T]$ is in the jth position. Hence

$$\begin{aligned}[D, [D, T^k]] &= \sum_{i,j=1}^{k} T \cdots T[D, T]T \cdots T \\ &= 2 \sum_{1\leq i<j\leq k} T \cdots T[D, T]T \cdots T[D, T]T \cdots T,\end{aligned} \tag{4.6}$$

where $[D, T]$ occupies the ith and jth positions. (This is a result of $\mathcal{N}$ being abelian and normal.) The k-fold bracket

$$[D, [D, \ldots [D, T^k] \cdots]] = k![D, T]^k, \tag{4.7}$$

while the $(k+1)$-fold bracket vanishes. By (1.8) of [C-R1, p.353],

$$2\pi i T^k f_{\alpha\beta}(t) = h_k(a\alpha\lambda^t + b\beta\lambda^{-t})^{-(k+1)} \tag{4.8}$$

where

$$\begin{aligned} h_k = {} & [D[D \cdots [D, T^k] \cdots]]g_{\alpha\beta} + D[D \cdots [D, T^k] \cdots]g_{\alpha\beta} \\ & + \cdots + D^{k-1}[D, T^k]g_{\alpha\beta} + D^k T^k g_{\alpha\beta}. \end{aligned}$$

It follows that $\|T^k f_{\alpha\beta}\|_2^2 = \int_{|t-t_0|\leq 1} + \int_{|t-t_0|>1}$, with the integrand being determined by (4.8).

The second integral can be estimated as in the case of $k = 1$, while the denominator in the integrand in the first integral has a Taylor expansion in $(t-t_0)$ using derivatives of order $\leq k+1$. For $k > 1$ and odd, the $(k+1)$th derivative of $(a\alpha\lambda^t + b\beta\lambda^{-t})^{k+1}$ is of the form $(d/dt)^{k+1}[(a\alpha\lambda^t + b\beta\lambda^{-t})^{k+1}] = \sum_{j=0}^{(k+1)/2} c_j(a\alpha\lambda^t + b\beta\lambda^{-t})^{k+1-2j}(a\alpha\lambda^t - b\beta\lambda^{-t})^{2j} \ln^k \lambda$ with all $c_j \geq 0$ and $c_{(k+1)/2} > 0$. Thus, if $k+1$ is even,

$$\frac{d^{k+1}}{dt^{k+1}}(a\alpha\lambda^t + b\beta\lambda^{-t})^{k+1} \geq C(a\alpha\lambda^t - b\beta\lambda^{-t})^{k+1}, \qquad \text{where } C > 0.$$

If $k+1$ is odd, then

$$\frac{d^{k+1}}{dt^{k+1}}(a\alpha\lambda^t + b\beta\lambda^{-t})^{k+1} \geq C'|a\alpha\lambda^t - b\beta\lambda^{-t}|^{k+1}, \qquad \text{for } C' > 0,$$

by a similar calculation. Hence the minima over $|t-t_0| \leq 1$ in the resulting estimates proceed as in the case of $k=1$.

Case 3. $D \in \mathcal{N}^C$. If $D \in C \cdot X$ or $D \in C \cdot Y$, then $D \in C \cdot \mathcal{N}$ and is covered by Case 2. Otherwise, dividing by a constant, we can assume that $D = X + (a+ib)Y$, where $a, b \in R$. Thus

$$f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/2\pi(i(\alpha\lambda^t + a\beta\lambda^{-t}) - b\beta\lambda^{-t}). \tag{4.9}$$

Since it suffices to prove L^2-summability for the parts of $f_{\alpha\beta}$ corresponding to the real and imaginary parts of $g_{\alpha\beta}$ separately, we can assume $g_{\alpha\beta}$ is real in (4.9). Suppose also $b \neq 0$, since Case 2 would apply if b were 0. We note that $|f_{\alpha\beta}(t)| \leq |g_{\alpha\beta}(t)/2\pi b\beta\lambda^{-t}|$. The methods of Case 2 can be applied to prove $\sum f_{\alpha\beta} \in L^2$.

If $U \in \mathcal{U}(\mathcal{N})$, then $[D, U] = 0$, so that $\sum U f_{\alpha\beta} \in L^2$. We need to prove $\sum T^k f_{\alpha\beta} \in L^2$ for each fixed $k \in N$.

We begin with $k=1$. Then $Tf_{\alpha\beta}(t)$ is the derivative of the right side of (4.9). In the numerator, we get various derivatives of $g_{\alpha\beta}$, while the modulus of the denominator exceeds $4\pi^2b^2\beta^2\lambda^{-2t}$. The λ^{-2t} can be moved to the numerator as a derivation, and $\sum T f_{\alpha\beta} \in L^2$. For $k > 1$, similar reasoning applies.

Case 4. $D = T + i(aX + bY)$, $a, b \in R$. Up to isomorphism, we could assume that $a = b = 1$, except for the case in which either $a=0$ or $b=0$. Since $T+iX$ and $T+iY$ are very similar, we need to treat only the cases $T+iX$ and $T+i(X+Y)$. Since $T+i(X+Y)$ is more complicated, we will treat this case in detail, providing brief remarks to cover the simpler case of $T+iX$.

Actually, to simplify the constants we suppose $D = T + i(X+Y)/2\pi$, so that

$$f_{\alpha\beta}(t) = \left(\int_0^t g_{\alpha\beta}(x)\exp(-(\alpha\lambda^x + \beta\lambda^{-x})/\ln\lambda)\,dx + C\right)\exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda).$$

Subcase 4a. $\alpha > 0$ *and* $\beta > 0$. In this case $\exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda) \to \infty$ as $t \to \pm\infty$ (or in case $b=0$, $\exp(\alpha\lambda^t/\ln\lambda) \to \infty$ as $t \to \infty$ and $\to 1$ as $t \to -\infty$). Thus

$$\begin{aligned} f_{\alpha\beta}(t) &= -\int_t^\infty g_{\alpha\beta}(x)\exp((-(\alpha\lambda^x + \beta\lambda^{-x}) + (\alpha\lambda^t + \beta\lambda^{-t}))/\ln\lambda)\,dx \\ &= \int_{-\infty}^t \cdots dx. \end{aligned} \tag{4.10}$$

If $b=0$, the $\beta\lambda^{-x}$ and $\beta\lambda^{-t}$ terms do not appear in (4.10) and we simply use $t_{\alpha\beta} = 0$ in what follows. Also, the restriction $\beta > 0$ is not necessary when we deal with the $b=0$ case.

Note that $\frac{d}{dx}(\alpha\lambda^x + \beta\lambda^{-x}) = (\alpha\lambda^x - \beta\lambda^{-x})\ln\lambda = 0$ iff $x = \frac{1}{2}\log_\lambda(\beta/\alpha)$, which we denote henceforth by $-1 \leq t_{\alpha\beta} \leq 0$. Since α and $\beta > 0$, $\alpha\lambda^x + \beta\lambda^{-x}$ is a decreasing function on $(-\infty, t_{\alpha\beta})$ and an increasing function on $(t_{\alpha\beta}, \infty)$. By using the *first* integral in (4.10) whenever $t \geq t_{\alpha\beta}$ and the *second* integral whenever $t < t_{\alpha\beta}$, we can assure that the exponential function in the integrand remains bounded by 1. In either case, $|g_{\alpha\beta}|$ can be bounded by one of its (X or Y) derivatives times an exponential function, with the result that $\sum f_{\alpha\beta} \in L^2$

of $(-\infty, t_{\alpha\beta}) \cup [t_{\alpha\beta}, \infty) = (-\infty, \infty)$. (If $b = 0$, use $(X^2 + Y^2)g$ instead.) Restriction to $\lambda^{-2} \le \beta/\alpha \le 1$ will keep α^{-1} bounded in absolute value.

Next, let $U = (2\pi)^{-n-m} X^m Y^n \in \mathscr{U}(\mathscr{N})$. If $m - n > 0$ and $t > t_{\alpha\beta}$, then

$$|Uf_{\alpha\beta}(t)| \le \left| \alpha^m \beta^n \lambda^{(m-n)t} \int_t^\infty |g_{\alpha\beta}(x)|\, dx \right| \le \int_t^\infty |Ug_{\alpha\beta}(x)|\, dx,$$

which provides the necessary estimate. If $m - n < 0$ and $t > t_{\alpha\beta}$,

$$|Uf_{\alpha\beta}(t)| \le |\alpha^m \beta^n| \lambda^{(m-n)t} \int_t^\infty |g_{\alpha\beta}(x)|\, dx.$$

But $\lambda^{-2} \le \beta/\alpha \le 1$ implies $t_{\alpha\beta} \ge -1$, so

$$\begin{aligned} |Uf_{\alpha\beta}(t)| &\le \lambda^{n-m} \int_t^\infty |\alpha^m \beta^n g_{\alpha\beta}(x)|\, dx \le \lambda^{n-m} \int_t^\infty |\alpha^{m+n} g_{\alpha\beta}(x)|\, dx \\ &= \lambda^{n-m} \int_t^\infty |X^{m+n} g_{\alpha\beta}(x)| \lambda^{-(m+n)x} (2\pi)^{-m-n}\, dx \\ &\le \lambda^{2n} (2\pi)^{-m-n} \int_t^\infty |X^{m+n} g_{\alpha\beta}(x)|\, dx. \end{aligned}$$

From here the L^2-estimates proceed as earlier.

Next, suppose $m - n < 0$ and $t < t_{\alpha\beta}$. Then $\lambda^{(m-n)t} > \lambda^{(m-n)t_{\alpha\beta}}$. Thus

$$\begin{aligned} |Uf_{\alpha\beta}(t)| &\le |\alpha^m \beta^n| \lambda^{(m-n)t} \int_{-\infty}^t |g_{\alpha\beta}(x)|\, dx \le \int_{-\infty}^t |\alpha^m \beta^n \lambda^{(m-n)x} g_{\alpha\beta}(x)|\, dx \\ &= \int_{-\infty}^t |Ug_{\alpha\beta}(x)|\, dx. \end{aligned}$$

The rest is as before.

Finally, for $m - n \ge 0$ and $t < t_{\alpha\beta}$, we write

$$\begin{aligned} |Uf_{\alpha\beta}(t)| &\le |\alpha^m \beta^n| \lambda^{(m-n)t} \int_{-\infty}^t |g_{\alpha\beta}(x)|\, dx \\ &\le |\lambda^{2m} \beta^m \beta^n| \lambda^{(m-n)t_{\alpha\beta}} \int_{-\infty}^t |g_{\alpha\beta}(x)|\, dx \\ &\le \lambda^{2m} \int_{-\infty}^t |\beta^{m+n} g_{\alpha\beta}(x)|\, dx \\ &= \int_{-\infty}^t |Y^{m+n} g_{\alpha\beta}(x)| \lambda^{(m+n)x} / (2\pi)^{m+n}\, dx \\ &\le (2\pi)^{-m-n} \int_{-\infty}^t |Y^{m+n} g_{\alpha\beta}(x)|\, dx. \end{aligned}$$

The L^2-estimates can be completed as before.

Next, we show that $\sum_{(\alpha, \beta)} T^k f_{\alpha\beta} \in L^2$, for each $k \in N$. This follows from the next lemma.

(4.11) **Lemma.** *Let f be a solution of the equation*

$$(T + i(X + Y))f = g.$$

Then $T^k f$, $k = 1, 2, 3, \ldots$, *is a linear combination of monomials* $X^j Y^l f$ *with* $j + l \le k$ *plus a linear combination of* X, Y, T*-derivatives of* g.

Proof. We proceed by induction. For $k = 1$ we have $Tf = g - iXf - iYf$. Next, $T^{k+1} f = T(T^k f) = T(X^j Y^l f)$ with $j + l \le k$, where wlog we may asume $T^k f$ is a monomial $X^j Y^l f$.

$$\begin{aligned} T(X^j Y^l f) &= [T, X^j Y^l] f + Y^j Y^l T f \\ &= \left(\sum_{p=1}^{j} X \cdots X[T, X] X \cdots X Y^l + \sum_{q=1}^{l} X^j Y \cdots Y[T, Y] Y \cdots Y \right) f \\ &\quad + Y^j Y^l g - i X^{j+1} Y^l f - i X^j Y^{l+1} f \\ &= (j \ln \lambda X^{j+1} Y^l - l \ln \lambda X^j Y^{l+1}) f + \ldots \end{aligned}$$

which is the desired expression for $T^k f$.

Remark. Similarly, $T^k f$ is a linear combination of monomials $X^j f$ with $j \le k$ plus a linear combination of X, T-derivatives of g if f is a solution of $(T + iX) f = g$.

Subcase 4b. $\alpha < 0$ *and* $\beta > 0$. (The case $\alpha > 0$ and $\beta < 0$ can be treated similarly.) Once again, we have

$$f_{\alpha\beta}(t) = \left(\int_0^t g_{\alpha\beta}(x) \exp(-(\alpha\lambda^x + \beta\lambda^{-x})/\ln\lambda)\, dx + C \right) \exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda)$$

where the terms $\beta\lambda^{-x}$ and $\beta\lambda^{-t}$ are not present if $b = 0$. Moreover, the restriction $\beta > 0$ is not needed if $b = 0$. We observe $\exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda) \to 0$ as $t \to +\infty$ and $\to +\infty$ as $t \to -\infty$ (or $\to 1$ as $t \to -\infty$ in case $b = 0$). In either case, since $\lim_{t\to-\infty} f_{\alpha\beta}(t) = 0$,

$$C = \int_{-\infty}^0 g_{\alpha\beta}(x) \exp(-(\alpha\lambda^x + \beta\lambda^{-x})/\ln\lambda)\, dx$$

and

$$f_{\alpha\beta}(t) = \int_{-\infty}^t g_{\alpha\beta}(x) e^{\psi(x,t)}\, dx, \tag{4.12}$$

where

$$\psi(x, t) = (\alpha(\lambda^t - \lambda^x) + \beta(\lambda^{-t} - \lambda^{-x}))/\ln\lambda,$$

again with no $\beta(\lambda^{-t} - \lambda^{-x})$ term in case $b = 0$. We notice that $\psi(x, t) < 0$ for $x < t$. We have the estimates

$$\begin{aligned} |f_{\alpha\beta}(t)| &\le \int_{-\infty}^t |g_{\alpha\beta}(x)|\, dx \le \int_{-\infty}^t |Y g_{\alpha\beta}(x)| / |2\pi\beta\lambda^{-x}|\, dx \\ &\le C \sum_{k=0}^{1} \|T^k Y g_{\alpha\beta}\|_2 |\beta|^{-1} \int_{-\infty}^t \lambda^x\, dx \quad \text{(by Sobolev)} \qquad (4.13) \\ &\le M \sum_{k=0}^{1} \|T^k Y g_{\alpha\beta}\|_2 \lambda^t / \ln\lambda \in L^2(-\infty, 0) \end{aligned}$$

since $|\beta|^{-1}$ is bounded. In fact, (4.13) implies that $\sum_{(\alpha,\beta)} f_{\alpha\beta} \in L^2(-\infty, 0)$. Next, we must consider convergence in $L^2(0, \infty)$. Thus for $t > 0$ we write

$$\begin{aligned} f_{\alpha\beta}(t) &= \int_{-\infty}^{0} \cdots + \int_{0}^{t/2} \cdots + \int_{t/2}^{t} \cdots \\ &= \mathrm{I}_{\alpha\beta}(t) + \mathrm{II}_{\alpha\beta}(t) + \mathrm{III}_{\alpha\beta}(t) \end{aligned}$$

where the integrands are as in (4.12).

To estimate $\mathrm{I}_{\alpha\beta}$ we notice that for $x \le 0 \le t$ we have $\beta(\lambda^{-t} - \lambda^{-x}) \le 0$ and $\alpha(\lambda^t - \lambda^x) < \alpha(\lambda^t - 1) < 0$. Hence

$$|\mathrm{I}_{\alpha\beta}(t)| \le e^{\alpha(\lambda^t - 1)/\ln\lambda} \int_{-\infty}^{0} |g_{\alpha\beta}(x)|\, dx \tag{4.14}$$

and $\sum_{(\alpha,\beta)} \mathrm{I}_{\alpha\beta} \in L^2(0, \infty)$. This is because $\alpha \gg 0$ makes the functions

$$t \mapsto \exp(\alpha(\lambda^t - \lambda^x)/\ln\lambda), \qquad (\alpha, \beta) \in (\Gamma\backslash S_h)\hat{},$$

uniformly $L^2(0, \infty)$, while the integral $\int_{-\infty}^{0} \cdots$ in (4.14) can be estimated as in (4.13) making the sum finite.

For $\mathrm{II}_{\alpha\beta}$ we have the estimate

$$\begin{aligned} |\mathrm{II}_{\alpha\beta}(t)| &\le \int_{0}^{t/2} |g_{\alpha\beta}(x)|\, e^{\alpha(\lambda^t - \lambda^x)/\ln\lambda}\, dx \\ &\le \|g_{\alpha\beta}\|_\infty \tfrac{t}{2}\, e^{\alpha(\lambda^t - \lambda^{t/2})/\ln\lambda}. \end{aligned} \tag{4.15}$$

The right-hand side again is α-uniformly in $L^2(0, \infty)$ with $\|g_{\alpha\beta}\|_\infty$ being (α, β)-summable.

Finally,

$$\begin{aligned} |\mathrm{III}_{\alpha\beta}(t)| &\le \int_{t/2}^{t} |g_{\alpha\beta}(x)|\, dx \\ &\le \int_{t/2}^{t} |X^m g_{\alpha\beta}(x)|/|2\pi\alpha\lambda^x|^m\, dx \\ &\le \|X^m g_{\alpha\beta}\|_\infty \tfrac{t}{2} M^m \lambda^{-mt}, \end{aligned} \tag{4.16}$$

where M is an upper bound on $|\alpha|^{-1}$, $t\lambda^{-mt} \in L^2(0, \infty)$, and $\|X^m_{\alpha\beta}\|^2_\infty$ is (α, β)-summable.

Next, we must show $\sum_{(\alpha,\beta)} U f_{\alpha\beta} \in L^2(R)$ for every fixed $U \in \mathscr{U}(\mathscr{S}_h)$.

If $U = Y^k$ we have the estimate

$$|Y^k f_{\alpha\beta}(t)| \le \int_{-\infty}^{t} |Y^k g_{\alpha\beta}(x)|\, e^{\psi(x,t)}\, dx. \tag{4.17}$$

As in the beginning of Subcase 4b we can show that $\sum Y^k f_{\alpha\beta} \in L^2(R)$.

For $U = X^k$, $\sum X^k f_{\alpha\beta} \in L^2(-\infty, 0)$ because for $t \le 0$,

$$|X^k f_{\alpha\beta}(t)| = \lambda^{2kt} |\alpha/\beta|^k |Y^k f_{\alpha\beta}(t)| \le \lambda^2 |Y^k f_{\alpha\beta}(t)| \tag{4.18}$$

if we choose (α, β) such that $\lambda^{-2} \le |\beta/\alpha| \le 1$. If $t > 0$, we consider $X^k \mathrm{I}_{\alpha\beta}$, $X^k \mathrm{II}_{\alpha\beta}$, and $X^k \mathrm{III}_{\alpha\beta}$ and we get the estimates (4.14), (4.15), and (4.16), each multiplied by λ^{2kt} and with $g_{\alpha\beta}$ replaced by $Y^k g_{\alpha\beta}$, as it was done in (4.18).

Finally, let $U = X^p Y^q T^r$. Case of $r \geq 1$ reduces to $r = 0$ by the Lemma (4.11). If $r = 0$, we apply X^p to $\int_{-\infty}^t |Y^q g_{\alpha\beta}(x)| e^{\psi(x,t)}\,dx$ as we applied $U = X^k$ to $\int_{-\infty}^t g_{\alpha\beta}(x)\, e^{\psi(x,t)}\,dx$.

Subcase 4c. $\alpha < 0$ and $\beta < 0$. We have

$$f_{\alpha\beta}(t) = \left(\int_0^t g_{\alpha\beta}(x)\, e^{-(\alpha\lambda^x + \beta\lambda^{-x})/\ln\lambda}\,dx + C\right) e^{(\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda}. \tag{4.19}$$

The function $Ce^{(\alpha\lambda^t+\beta\lambda^{-t})/\ln\lambda}$ is in $\mathscr{S}\mathscr{S}(R)$ if $\alpha < 0$ and $\beta < 0$. Hence if there is a constant C such that $f_{\alpha\beta}$ in (4.19) is super-Schwartz, then $f_{\alpha\beta} \in \mathscr{S}\mathscr{S}(R)$ for any fixed C. We will pick $C = C_{\alpha\beta} = \int_{t_{\alpha\beta}}^0 g_{\alpha\beta}(x)\, e^{-(\alpha\lambda^x+\beta\lambda^{-x})/\ln\lambda}\,dx$, where $t_{\alpha\beta} := \frac{1}{2}\log_\lambda(\beta/\alpha)$, and α, β are chosen so that $\lambda^{-2} < \beta/\alpha \leq 1$. Thus we will work with

$$f_{\alpha\beta}(t) = \left(\int_{t_{\alpha\beta}}^t g_{\alpha\beta}(x)\, e^{-(\alpha\lambda^x + \beta\lambda^{-x})/\ln\lambda}\,dx\right) e^{(\alpha\lambda^t + \beta\lambda^{-t})/\ln\lambda}. \tag{4.19$'$}$$

We have the estimate

$$\begin{aligned} |f_{\alpha\beta}(t)| &\leq e^{(\alpha\lambda^t+\beta\lambda^{-t})/\ln\lambda} \int_{t_{\alpha\beta}}^t (2\pi)^{-m} |(X+Y)^m g_{\alpha\beta}(x)| \\ &\qquad \times e^{-(\alpha\lambda^x+\beta\lambda^{-x})/\ln\lambda}(-\alpha\lambda^x - \beta\lambda^{-x})^{-m}\,dx \\ &\leq e^{\cdots/\ln\lambda}\|(X+Y)^m g_{\alpha\beta}\|_\infty e^{-\cdots/\ln\lambda}(-2\pi\cdots)^{-m}|t - t_{\alpha\beta}| \\ &\leq c\sum_{l=0}^{1} \|T^l (X+Y)^m g_{\alpha\beta}\|_2 (-\alpha\lambda^t - \beta\lambda^{-t})^{-m}|t - t_{\alpha\beta}| \end{aligned} \tag{4.20}$$

where $\cdots$ stands for $\alpha\lambda^t + \beta\lambda^{-t}$. We write the inequality (4.20) for (α, β) such that

$$2(\alpha\beta)^{1/2}/\ln\lambda > m, \tag{4.21}$$

because then the function $u \mapsto e^{-(\alpha u+\beta u^{-1})/\ln\lambda}(-\alpha u - \beta u^{-1})^{-m}$ with $u = \lambda^t$ is increasing for $t \geq t_{\alpha\beta}$ and decreasing for $t \leq t_{\alpha\beta}$. (4.21) is valid for all but a finite number of $(\alpha, \beta) \in (\Gamma\backslash S_h)\hat{}_\infty$. Similarly, for $Y^l X^k f_{\alpha\beta}$ we have the estimate

(4.22)

$$|Y^l X^k f_{\alpha\beta}(t)| \leq c_1 \sum_{p=0}^{1} \|T^p (X+Y)^m g_{\alpha\beta}\|_2 |\beta\lambda^{-t}|^l |\alpha\lambda^t|^k (-\alpha\lambda^t - \beta\lambda^{-t})^{-m} |t - t_{\alpha\beta}|.$$

But for $t \geq t_{\alpha\beta}$, if $m > k + l$

$$\begin{aligned} |\beta\lambda^{-t}|^l |\alpha\lambda^t|^k (-\alpha\lambda^t - \beta\lambda^{-t})^{-m} |t - t_{\alpha\beta}| &\leq (\beta/\alpha)^l (-\alpha)^{k+l-m} \lambda^{t(k-l-m)} |t - t_{\alpha\beta}| \\ &\leq M^{m-k-l} |t - t_{\alpha\beta}| \lambda^{t(k-l-m)} \in L^2(0, \infty) \end{aligned}$$

since $\beta/\alpha \leq 1$ and $\alpha \gg 0$.

Similarly, for $t \leq t_{\alpha\beta}$

$$\begin{aligned} |\beta\lambda^{-t}|^l |\alpha\lambda^t|^k (-\alpha\lambda^t - \beta\lambda^{-t})^{-m} |t - t_{\alpha\beta}| &\leq (\alpha/\beta)^k (-\beta)^{k+l-m} \lambda^{t(m-l+k)} |t_{\alpha\beta} - t| \\ &\leq \lambda^{2k} M^{m-k-l} |t - t_{\alpha\beta}| \lambda^{t(m-l+k)} \in L^2(-\infty, 0) \end{aligned}$$

since $\lambda^{-2} \leq \beta/\alpha$ and $\beta \gg 0$. Thus $\sum Y^l X^k f_{\alpha\beta} \in L^2(R)$. Finally,

$$\sum Y^l X^k T^m f_{\alpha\beta} \in L^2(R)$$

by Lemma (4.11).

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